(Boundary) regularity for mass minimizing currents

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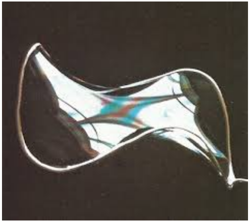
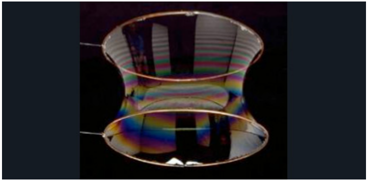
The Plateau Problem

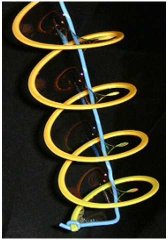
The Plateau Problem is named after the Belgian physicist **Joseph Plateau** (1801-1883) who was interested in the study of *soap bubbles*.

The classical Plateau Probelm

Given a curve Γ in $\mathbb{R}^3$ find a *surface* of minimal *area* which *spans* Γ.







A general formulation of the Plateau problem

The generalised Plateau Problem

Given a $(m-1)$ dimensional manifold Γ in a n-dimensional Riemannian manifold $\mathcal{M}^n$ $(m < n)$ find a m-dimensional surface $\Sigma \subset \mathcal{M}$ of minimal "area" (m-dimensional volume) spanning Γ $(\partial\Sigma = \Gamma)$.

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$$\Downarrow$$

Geometric Measure Theory.

The direct methods in the Calculus of Variations

Let $\{\Sigma_j\}$ be a minimising sequence, i.e.

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$$\text{Area}(\Sigma_\infty) \leq \liminf \text{Area}(\Sigma_j)$$

Indeed in this case

$$\text{Area}(\Sigma_\infty) \leq \liminf \text{Area}(\Sigma_j) = \inf \Big\{ \text{Area}(\Sigma) : \partial\Sigma = \Gamma \Big\}.$$

and Σ_∞ is admissible.

The direct methods in the Calculus of Variations

Three possible approaches:

Parametrized approach: Douglas, Rado, Courant,...

Set theoretical approach: Reifenberg, Almgren, Harrison-Pugh, De Lellis-Ghiraldin-Maggi, D.-De Rosa-Ghiraldin,...

Distributional approach: De Giorgi, Federer-Fleming,...

The parametrized approach

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Let $\Gamma \subset \mathcal{M}^n$ be a *Jordan curve*, i.e. $\Gamma = \varphi(\mathbb{S}^1)$, φ injective and continuous. The class of admissible surfaces is given by *images* of maps from the unit disk $\mathbb{D} \subset \mathbb{R}^2 \approx \mathbb{C}$ such that

$$X(\partial\mathbb{D}) \subset \Gamma$$

and

$$X : \partial\mathbb{D} \to \Gamma \quad \text{is a weakly monotone parametrization.}$$

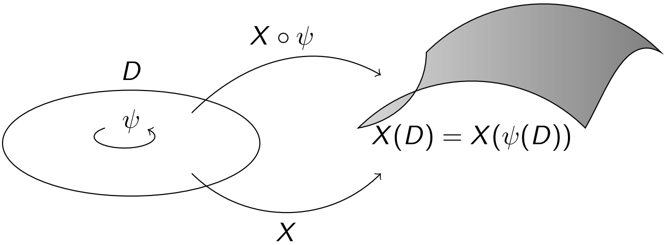
(Note that we are not imposing that $X\Big|_{\partial\mathbb{D}} = \varphi$)

The parametrized approach

The area functional

$$\mathrm{Area}(X) = \int_{\mathbb{D}} \left|\partial_x X \wedge \partial_y X\right|.$$

is invariant under reparamerization:



If $\psi : D \to D$ is a diffeomorphism

$$\mathrm{Area}(X) = \mathrm{Area}(X \circ \psi)$$

but possibly $\|X \circ \psi\| \gg \|X\|$, $\Rightarrow$ no control on the parametrization!

The parametrized approach

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However:

$$|\partial_x X \wedge \partial_y X| \leq |\partial_x X||\partial_y X| \leq \frac{|\partial_x X|^2 + |\partial_y X|^2}{2}.$$

so that

$$\text{Area}(X) \leq \text{Energy}(X) := \frac{1}{2}\int_{\mathbb{D}} |\nabla X|^2.$$

Moreover we have equality if (and only if) X is *conformal*:

$$|\partial_x X| = |\partial_y X| \qquad \partial_x X \cdot \partial_y X = 0.$$

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Theorem (Douglas-Rado)

There exists a conformal minimizer $\bar{X}$ of Energy. Furthermore

$$\text{Area}(\bar{X}) = \inf\Big\{\text{Area}(X) : \\ X : \mathbb{D} \to \mathcal{M}^n, \quad X : \partial\mathbb{D} \to \Gamma \quad \text{monotone parametrization}\Big\}$$

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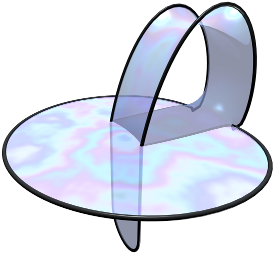

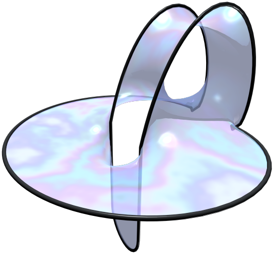

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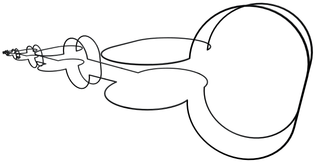

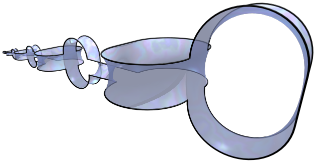

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This is the good framework to study soap bubbles!

The distributional approach

Let Σ be a smooth m-dimensional surface, then

$$\mathcal{D}^m(\mathcal{M}^n) \ni \omega \mapsto [\![\Sigma]\!](\omega) := \int_\Sigma \omega$$

is a continuous linear functional on the space of compactly supported smooth m-dimensional forms.

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We can recover the geometric data of Σ by its action on forms!

Currents

Definition (De Rahm/Federer-Fleming)

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- Convergence:

$$T_j \overset{*}{\rightharpoonup} T \qquad \iff \qquad T_j(\omega) \to T(\omega) \qquad \forall \omega.$$

The Plateau problem with currents

By abstract non-sense (Banach-Alouglu Theorem) we have:

Theorem

Given a $(m-1)$ dimensional manifold Γ in a n-dimensional Riemannian manifold $\mathcal{M}^n$ there exists m-dimensional current T with $\operatorname{spt} T \subset \mathcal{M}^n$ such that

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The problem is that we added too many competitors!

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Let

$$\omega = a(x,y)dx + b(x,y)dy \in \mathcal{D}^1(\mathbb{R}^2)$$

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Question

Can the above examples arise as limit of a minimising sequence of the original Plateau problem?

Federer-Fleming Closure Theorem

Theorem (Federer-Fleming)

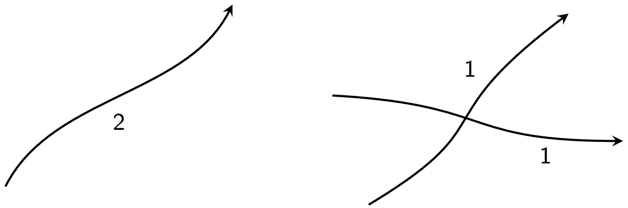
The weak-$*$ closure of

$$\left\{ [\![\Sigma]\!] : \ \Sigma \textit{ is a smooth } m\textit{-dim surface with } \partial\Sigma = \Gamma \textit{ and } \mathrm{Area}(\Sigma) \leq c \right\}$$

is given by the class of integer rectifiable currents*.*

Integer rectifiable currents

Integer rectifiable currents are *countably union of "pieces" of C^1 manifolds* with *integer* multiplicity.



Integer rectifiable currents

Definition

A m-dimensional current T is said to be integer rectifiable if there exist two sequences $\{K_j\}$ and $\{\theta_j\}$ such that

- K_j *is a compact subset of* C^1 *m-dimensional surface* M_j*,*
- $\theta_j \in \mathbb{N}$*,*
- $\sum_j \theta_j \mathrm{Area}(K_j) < +\infty$

and

$$T(\omega) = \sum_j \theta_j \int_{K_j} \omega.$$

The deformation Theorem

Theorem (Federer-Fleming)

The infimum among of the Plateau problem among smooth manifolds is equal to the minimum of the Plateau problem among integer rectifiable currents.

Regularity

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Note that this would allow to solve the problem in the smooth category.

In particular when $m = 2$ it would prove that that for all (smooth) Γ there exists g_0 such that

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Regularity divides into:

- Interior regularity (regularity away from Γ)
- Boundary regularity (regularity close to Γ)

Interior regular points

Definition

An interior *point* $p \in \operatorname{spt} T \setminus \Gamma$ *is regular,* $p \in \mathrm{Reg_i}(T)$*, if there exists a neighborhood* U *of* p *and a smooth manifold* Σ *such that*

$$T \llcorner U = Q[\![\Sigma]\!] \qquad \text{for some } Q \in \mathbb{N}.$$

Interior regularity

The regularity theory highly depends on the co-dimension $n - m$, let

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- **Co-dimension one** ($n = m + 1$): De Giorgi/Federer/Simons:

$$\dim_{\mathcal{H}} \mathrm{Sing_i}(T) \leq m - 7.$$

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- **High co-dimension** ($n \geq m + 2$): Almgren+De Lellis-Spadaro:

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- The current associated with the cone

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The proof of the two regularity results is quite different and Almgren's proof is 1000 pages long!

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- **Co-dimension one** ($m = 2$, $n = 3$): Minimizers are smooth away from Γ.
- **High co-dimension** ($m = 2$, $n \geq 4$),
  Chang+De Lellis-Spadaro-Spolaor: $\mathrm{Sing}_{\mathrm{i}}(T)$ is discrete and locally around $p \in \mathrm{Sing}_{\mathrm{i}}(T)$, spt T is given by finitely many branched disk intersecting at p.

Note that the second result is perfectly coherent with the structure of complex variety!

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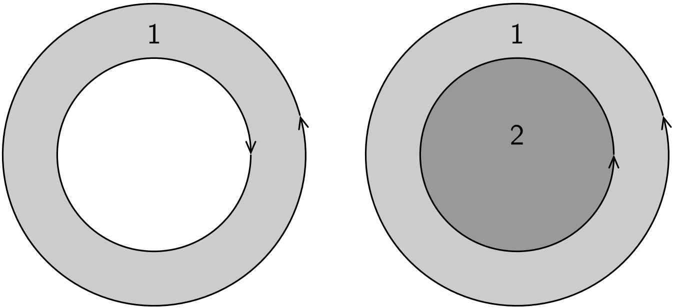

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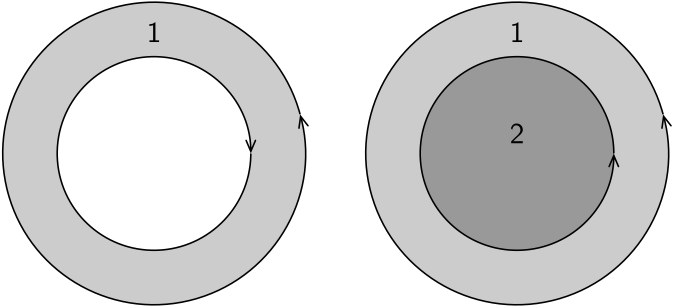


Note that there are boundary points which lies at the interior of spt T!

Boundary regular points

Definition

A boundary *point* $p \in \Gamma$ *is regular,* $p \in \mathrm{Reg_b}(T)$*, if there exists a neighborhood* U *of* p *and a smooth m-dimensional manifold* Σ *such that or some* $Q \in \mathbb{N}$*.*

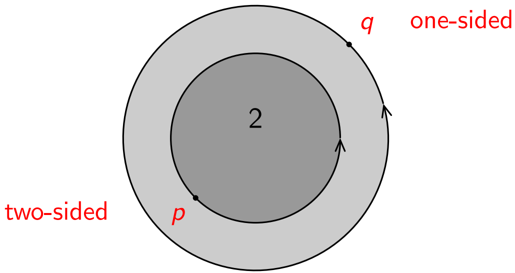
$$T \llcorner U = Q[\![\Sigma_+]\!] + (Q-1)[\![\Sigma_-]\!] \qquad \text{for some } Q \in \mathbb{N}.$$

where $\Sigma_\pm$ *are the two parts in which* Γ *splits* Σ*.*

We will say that

- p is a *regular one-sided point* if $Q = 1$;
- p is a *regular two-sided point* if $Q \geq 2$;

Back to the example...



Note that defining

$$\Theta(T, x) = \lim_{r \to 0} \frac{\boldsymbol{M}(T \llcorner B_r(x))}{\omega_m r^m},$$

then

$$\Theta(T, q) = \frac{1}{2} \qquad \Theta(T, p) = \frac{3}{2}$$

Interlude

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No, if there exists at least one regular boundary point, in particular the multiplicity of T is 1 almost everywhere (not too difficult to show).

Boundary regularity

Theorem (Allard)

One sided points are always regular, where

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Balls are convex and can be used as barriers:

$$q \in \operatorname{argmax}\{|p| : p \in \Gamma\} \quad \text{is one-sided.}$$

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Corollary

In co-dimension 1 there are no regular two sided points if Γ is connected (and smooth).

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When the co-dimension is ≥ 2 it is not known in a general ambient manifold if there exists *one* boundary regular point (and if the ambient is $\mathbb{R}^n$ only the existence of very few ones is known).

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There exists a two-dimensional mass minimising current with a sequence of singular points accumulating at the boundary.

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Moreover (compare with Hardt-Simon's corollary)

Theorem (De Lellis-D.-Hirsch)

There exists a smooth 4 dimensional Riemannian manifold and a smooth curve Γ such that the mass minimizing current spanned by Γ has infinite topology.

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Theorem (DDHM)

Collapsed points are always regular.

Thank you!